Int. J. Solids Structures Vol. 26, No. 3, pp. 255–270, 1990
Printed in Great Britain.
0020–7683/90 $3.00+ .00

ANALYSIS OF FREE VIBRATIONS OF NONCIRCULAR THICK CYLINDRICAL SHELLS HAVING CIRCUMFERENTIALLY VARYING THICKNESS

K. SUZUKI
Faculty of Engineering, Yamagata University, Yonezawa, Japan

and

A. W. LEISSA†
Department of Engineering Mechanics, United States Air Force Academy, Colorado Springs, Colorado, U.S.A.

(*Received* 23 *June* 1988; *in revised form* 3 *May* 1989)

Abstract—A method, based on the improved thick shell theory of Mirsky and Herrmann for circular thick cylindrical shells, is presented for analyzing the free vibrations of noncircular thick cylindrical shells having circumferential thickness variation. The equations of motion are solved exactly by using a power series expansion. Frequencies and mode shapes are presented for elliptical cylindrical shells having second degree thickness variation. The effects of shear deformation and rotary inertia upon the frequencies are discussed by comparing results from the present theory with those of thin shell theory.

1. INTRODUCTION

There exist a large number of references that deal with the vibrations of axisymmetric shells of revolution (cf. Leissa, 1973). Recently, some studies on the vibrations of noncircular cylindrical shells such as an elliptical cylindrical shell or an oval shell have been made (cf. Suzuki *et al.*, 1983), but most of these works treat shells having uniform thickness. The present authors (Suzuki and Leissa, 1985) studied the free vibrations of noncircular cylindrical shells having circumferentially varying thickness by using thin shell theory. They developed an exact solution procedure for determining the free vibration frequencies and mode shapes of noncircular cylindrical shells having circumferentially varying thickness and the two opposite, curved edges supported by shear diaphragms (also called "freely supported ends").

The purpose of the present work is to present a set of governing equations and a method for analyzing the free vibrations of noncircular thick cylindrical shells having circumferentially varying thickness. Mirsky and Herrmann (1957) and Mirsky (1964) have given an excellent improved thick shell theory for investigating the vibrations of circular thick cylindrical shells. The present work is an extension to noncircular cylindrical shells of their improved thick shell theory for circular thick cylindrical shells. Equations of motion and boundary conditions that include the effects of shear deformation and rotary inertia are derived, and the equations of motion are solved exactly by using power series expansions. The method is demonstrated for a set of elliptical shells having quadratic thickness variation and both ends supported by shear diaphragms. Numerical results obtained by the present method are compared with those determined from thin shell theory, which were previously given by Suzuki and Leissa (1985), and the effects of shear deformation and rotary inertia upon the natural frequencies are discussed.

2. ANALYSIS

2.1. *Lagrangian formulation of equations of motion and boundary conditions*

Let us consider the free vibrations of a noncircular thick cylindrical shell for which the centerline of the cross-section is a smooth curve, and the thickness varies along the center-

† On leave from Ohio State University.

line, but is constant along the generator. This centerline, which is the intersection of the middle surface of the shell with the plane $x' = \text{const}$, is shown in Fig. 1. Let the length of the shell be $l = \mu r_0$, where r_0 is a representative radius of curvature parameter, and let the curvature at any point along the centerline be $1/\rho$. Take the coordinate axis x' along the generator of the middle surface, the arc length s measured along the centerline of the cross-section (the centerline and the generator being orthogonal) and the z-axis towards the center of curvature. Let the thickness of the shell be $h = h_0 H(s)$, where h_0 is the thickness at $s = 0$ and $H(s)$ is a function of s. Employ a nondimensional coordinate $x = x'/r_0$ and denote the displacements in the x', s and z directions by $\bar{u}$, $\bar{v}$ and $\bar{w}$, respectively.

It is convenient to use the following transformation of variable:

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{\rho} = \frac{G}{r_0}\Phi(\theta) \tag{1}$$

where θ is a variable that describes an angle between the tangent to the centerline curve at the origin of s and the one at an arbitrary point on the centerline, G is a constant determined by the shape of the curve and $\Phi(\theta)$ is a function of θ. As an example, consider an ellipse, for which the equation is denoted by the rectangular coordinates (ξ_1, ξ_2) as $\xi_1 = a\cos\eta$, $\xi_2 = b\sin\eta$, in which $2a$, $2b$ and η are the major and minor axes and a parameter, respectively. Setting the representative radius r_0 and the ellipticity of the curve μ_0 as

$$r_0 = \sqrt{(a^2+b^2)/2}, \quad \mu_0 = (a^2-b^2)/(a^2+b^2), \tag{2}$$

one obtains

$$\tan\theta = \sqrt{(1+\mu_0)/(1-\mu_0)}\,\tan\eta, \quad \frac{1}{G} = 1-\mu_0^2, \quad \Phi(\theta) = (1+\mu_0\cos 2\theta)^{3/2}. \tag{3}$$

One can denote $\Phi(\theta)$ by a simple expression for a number of other curves as well (cf. Suzuki *et al.*, 1978).

Following the improved thick shell theory for circular cylindrical shells of Mirsky and Herrmann (1957) and Mirsky (1964), it is assumed that the displacements $\bar{u}$, $\bar{v}$ and $\bar{w}$ are represented approximately by

$$\bar{u} = \left[u + \frac{z}{r_0}\psi_x\right]\sin\omega t$$
$$\bar{v} = \left[v + \frac{z}{r_0}\psi_\theta\right]\sin\omega t$$
$$\bar{w} = w\sin\omega t \tag{4}$$

where ω and t denote the circular frequency and time, respectively, and u, v, w, ψ_x and ψ_θ are functions of x and θ. The normal and the shearing strain expressions at any point are

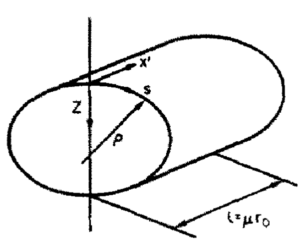

Fig. 1. Coordinates.

obtained from Love (1927):

$$\varepsilon_x = \frac{1}{r_0}\frac{\partial \bar{u}}{\partial x}, \qquad -\gamma_{\theta z} = \frac{1}{\rho - z}\left(\frac{\partial \bar{w}}{\partial \theta} + \bar{v}\right) + \frac{\partial \bar{v}}{\partial z}$$

$$\varepsilon_\theta = \frac{1}{\rho - z}\left(\frac{\partial \bar{v}}{\partial \theta} - \bar{w}\right), \quad -\gamma_{zx} = \frac{\partial \bar{u}}{\partial z} + \frac{1}{r_0}\frac{\partial \bar{w}}{\partial x}$$

$$\varepsilon_z = \frac{\partial \bar{w}}{\partial z}, \qquad \gamma_{x\theta} = \frac{1}{\rho - z}\frac{\partial \bar{u}}{\partial \theta} + \frac{1}{r_0}\frac{\partial \bar{v}}{\partial x}. \tag{5}$$

Substituting eqns (4) into eqns (5), one has

$$\varepsilon_x = \frac{1}{r_0}\left(\frac{\partial u}{\partial x} + \frac{z}{r_0}\frac{\partial \psi_x}{\partial x}\right)\sin \omega t, \quad \varepsilon_\theta = \frac{1}{\rho - z}\left(\frac{\partial v}{\partial \theta} - w + \frac{z}{r_0}\frac{\partial \psi_\theta}{\partial \theta}\right)\sin \omega t$$

$$\varepsilon_z = 0, \quad \gamma_{x\theta} = \left[\frac{1}{\rho - z}\left(\frac{\partial u}{\partial \theta} + \frac{z}{r_0}\frac{\partial \psi_x}{\partial \theta}\right) + \frac{1}{r_0}\left(\frac{\partial v}{\partial x} + \frac{z}{r_0}\frac{\partial \psi_\theta}{\partial x}\right)\right]\sin \omega t$$

$$-\gamma_{\theta z} = \left[\frac{1}{\rho - z}\left(v + \frac{\rho}{r_0}\psi_\theta + \frac{\partial w}{\partial \theta}\right)\right]\sin \omega t$$

$$-\gamma_{zx} = \left[\frac{1}{r_0}\left(\psi_x + \frac{\partial w}{\partial x}\right)\right]\sin \omega t. \tag{6}$$

Assuming a state of plane stress and that shearing stresses obey Hooke's law, one has

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\theta), \quad \tau_{x\theta} = \frac{E\gamma_{x\theta}}{2(1+\nu)}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\nu\varepsilon_x + \varepsilon_\theta), \quad \tau_{\theta z} = \frac{k' E\gamma_{\theta z}}{2(1+\nu)}$$

$$\sigma_z = 0, \qquad \tau_{zx} = \frac{k' E\gamma_{zx}}{2(1+\nu)} \tag{7}$$

where E, ν and k' are Young's modulus, Poisson's ratio and the shear coefficient, respectively. Let us now define the Lagrangian for a vibratory period (τ') as follows:

$$L = \left(\frac{2}{\tau'}\right)\frac{1}{2}\int_0^{\tau'}\int_{-h/2}^{h/2}\int_{\theta_1}^{\theta_2}\int_{x_1}^{x_2}\left[\rho_0\left\{\left(\frac{\partial \bar{u}}{\partial t}\right)^2 + \left(\frac{\partial \bar{v}}{\partial t}\right)^2 + \left(\frac{\partial \bar{w}}{\partial t}\right)^2\right\} - (\sigma_x\varepsilon_x + \sigma_\theta\varepsilon_\theta\right.$$

$$\left. + \sigma_z\varepsilon_z + \tau_{x\theta}\gamma_{x\theta} + \tau_{\theta z}\gamma_{\theta z} + \tau_{zx}\gamma_{zx})\right] r_0(\rho - z)\, \mathrm{d}x\, \mathrm{d}\theta\, \mathrm{d}z\, \mathrm{d}t \tag{8}$$

where ρ_0 denotes the mass density. Substituting eqns (1) and (4)–(7) into eqn (8) and using the following relations

$$\int_{-h/2}^{h/2}\frac{\mathrm{d}z}{\rho - z} \doteqdot \frac{h}{\rho}\left(1 + \frac{h^2}{12\rho^2}\right), \quad \int_{-h/2}^{h/2}\frac{z}{\rho - z}\,\mathrm{d}z \doteqdot \frac{h^3}{12\rho^2}$$

$$\int_{-h/2}^{h/2}\frac{z^2}{\rho - z}\mathrm{d}z \doteqdot \frac{h^3}{12\rho}, \qquad \int_{-h/2}^{h/2}\frac{z^3}{\rho - z}\,\mathrm{d}z \doteqdot \int_{-h/2}^{h/2}\frac{z^4}{\rho - z}\,\mathrm{d}z \doteqdot 0, \tag{9}$$

equation (8) becomes

$$-L\bigg/\left(\frac{D_0}{2r_0^2G}\right)=\int_{\theta_1}^{\theta_2}\int_{x_1}^{x_2}\Bigg(-\alpha_0^4H\left[u^2+v^2+w^2+\frac{H^2}{\beta_0}\{\psi_x^2+\psi_\theta^2-2G\Phi(u\psi_x+v\psi_\theta)\}\right]$$
$$+\beta_0H\left[\left\{\frac{\partial u}{\partial x}+G\Phi\left(\frac{\partial v}{\partial\theta}-w\right)\right\}^2-4\zeta G\Phi\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial\theta}-w\right)\right]$$
$$+H^3\left[\left(\frac{\partial\psi_x}{\partial x}\right)^2+G^2\Phi^2\left\{\frac{\partial\psi_\theta}{\partial\theta}+G\Phi\left(\frac{\partial v}{\partial\theta}-w\right)\right\}^2+2G\Phi\left\{(1-2\zeta)\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_\theta}{\partial\theta}-\frac{\partial u}{\partial x}\frac{\partial\psi_x}{\partial x}\right\}\right]$$
$$+\beta_0\zeta H\left(G\Phi\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial x}\right)^2+\zeta H^3\left[\left(\frac{\partial\psi_\theta}{\partial x}\right)^2+G^2\Phi^2\left(\frac{\partial\psi_x}{\partial\theta}+G\Phi\frac{\partial u}{\partial\theta}\right)^2\right.$$
$$\left.+2G\Phi\left(\frac{\partial\psi_x}{\partial\theta}\frac{\partial\psi_\theta}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial\psi_\theta}{\partial x}\right)\right]+k'\zeta\left[\beta_0H\left(\psi_x+\frac{\partial w}{\partial x}\right)^2\right.$$
$$\left.+(\beta_0H+G^2\Phi^2H^3)\left\{G\Phi\left(v+\frac{\partial w}{\partial\theta}\right)+\psi_\theta\right\}^2\right]\Bigg)\frac{\mathrm{d}x\,\mathrm{d}\theta}{\Phi} \tag{10}$$

where

$$\alpha_0^4=\frac{\rho_0h_0\omega^2r_0^4}{D_0},\quad \beta_0=\frac{12r_0^2}{h_0^2},\quad D_0=\frac{Eh_0^3}{12(1-v^2)},\quad \zeta=\frac{1-v}{2}. \tag{11}$$

Taking the first variation of L to obtain the stationary condition of the Lagrangian, $\delta L=0$, one obtains

$$\delta L\bigg/\left(\frac{D_0}{r_0^2G}\right)=\int_{\theta_1}^{\theta_2}\int_{x_1}^{x_2}[E_1\delta u+E_2\delta v+E_3\delta w+E_4\delta\psi_x+E_5\delta\psi_\theta]\,\mathrm{d}x\,\mathrm{d}\theta$$
$$-\int_{x_1}^{x_2}[T_1\delta u+T_2\delta v+T_3\delta w+M_1\delta\psi_x+M_2\delta\psi_\theta]_{\theta_1}^{\theta_2}\,\mathrm{d}x$$
$$-\int_{\theta_1}^{\theta_2}[T_4\delta u+T_5\delta v+T_6\delta w+M_3\delta\psi_x+M_4\delta\psi_\theta]_{x_1}^{x_2}\,\mathrm{d}\theta=0 \tag{12}$$

where

$$E_1=\alpha_0^4H\left[\frac{u}{\Phi}-\frac{GH^2}{\beta_0}\psi_x\right]+\frac{\partial T_1}{\partial\theta}+\frac{\partial T_4}{\partial x},$$
$$E_2=\alpha_0^4H\left[\frac{v}{\Phi}-\frac{GH^2}{\beta_0}\psi_\theta\right]-T_3+\frac{\partial T_2}{\partial\theta}+\frac{\partial T_5}{\partial x},$$
$$E_3=\alpha_0^4H\frac{w}{\Phi}+T_2+\frac{\partial T_3}{\partial\theta}+\frac{\partial T_6}{\partial x},$$
$$E_4=\frac{\alpha_0^4}{\beta_0}H^3\left[\frac{\psi_x}{\Phi}-Gu\right]-T_6+\frac{\partial M_1}{\partial\theta}+\frac{\partial M_3}{\partial x},$$
$$E_5=\frac{\alpha_0^4}{\beta_0}H^3\left[\frac{\psi_\theta}{\Phi}-Gv\right]-T_3/G\Phi+\frac{\partial M_2}{\partial\theta}+\frac{\partial M_4}{\partial x} \tag{13}$$

and where

$$T_1 = \zeta Gh\left[\beta_0\left(G\Phi\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}\right)+G^2H^2\Phi^2\left(\frac{\partial \psi_x}{\partial \theta}+G\phi\frac{\partial u}{\partial \theta}\right)\right],$$

$$T_2 = GH\left[\beta_0(1-2\zeta)\frac{\partial u}{\partial x}+(\beta_0+G^2H^2\Phi^2)G\Phi\left(\frac{\partial v}{\partial \theta}-w\right)+G^2H^2\Phi^2\frac{\partial \psi_\theta}{\partial \theta}\right],$$

$$T_3 = k'\zeta GH(\beta_0+G^2H^2\Phi^2)\left\{G\Phi\left(v+\frac{\partial w}{\partial \theta}\right)+\psi_\theta\right\},$$

$$T_4 = GH\left[\beta_0\left\{\frac{1}{G\Phi}\frac{\partial u}{\partial x}+(1-2\zeta)\left(\frac{\partial v}{\partial \theta}-w\right)\right\}-H^2\frac{\partial \psi_x}{\partial x}\right],$$

$$T_5 = \zeta H\left[\beta_0\left\{G\frac{\partial u}{\partial \theta}+\frac{1}{\Phi}\frac{\partial v}{\partial x}\right\}-GH^2\frac{\partial \psi_\theta}{\partial x}\right],$$

$$T_6 = k'\zeta\beta_0 H\left(\psi_x+\frac{\partial w}{\partial x}\right)\Big/\Phi,$$

$$M_1 = \zeta GH^3\left[G\Phi\left(\frac{\partial \psi_x}{\partial \theta}+G\Phi\frac{\partial u}{\partial \theta}\right)+\frac{\partial \psi_\theta}{\partial x}\right],$$

$$M_2 = GH^3\left[G\Phi\left\{\frac{\partial \psi_\theta}{\partial \theta}+G\Phi\left(\frac{\partial v}{\partial \theta}-w\right)\right\}+(1-2\zeta)\frac{\partial \psi_x}{\partial x}\right],$$

$$M_3 = H^3\left[\frac{1}{\Phi}\frac{\partial \psi_x}{\partial x}+G\left\{(1-2\zeta)\frac{\partial \psi_\theta}{\partial \theta}-\frac{\partial u}{\partial x}\right\}\right],$$

$$M_4 = \zeta H^3\left[\frac{1}{\Phi}\frac{\partial \psi_\theta}{\partial x}+G\left(\frac{\partial \psi_x}{\partial \theta}-\frac{\partial v}{\partial x}\right)\right]. \qquad (14)$$

Observing that

$$\{T_1, T_2, T_3\}D_0 \sin \omega t/r_0^3 G = \int_{-h/2}^{h/2} \{\tau_{x\theta}, \sigma_\theta, -\tau_{\theta z}\}\, dz$$

$$\{T_4, T_5, T_6\}D_0\Phi \sin \omega t/r_0^3 = \int_{-h/2}^{h/2} \{\sigma_x, \tau_{x\theta}, -\tau_{xz}\}(1-z/\rho)\, dz$$

$$\{M_1, M_2\}D_0 \sin \omega t/r_0^2 G = \int_{-h/2}^{h/2} \{\tau_{x\theta}, \sigma_\theta\}z\, dz$$

$$\{M_3, M_4\}D_0\Phi \sin \omega t/r_0^2 = \int_{-h/2}^{h/2} \{\sigma_x, \tau_{x\theta}\}z(1-z/\rho)\, dz, \qquad (15)$$

one may readily see that the quantities $(T_1, T_2, T_3)D_0/r_0^3G$, $(T_4, T_5, T_6)D_0\Phi/r_0^3$ are proportional to the resultant forces per unit length acting on the shell element and the quantities $(M_1, M_2)D_0/r_0^2G$, $(M_3, M_4)D_0\Phi/r_0^2$ to the resultant moments, respectively. Euler's equations (equations of motion) are

$$E_1 = E_2 = E_3 = E_4 = E_5 = 0. \qquad (16)$$

The boundary conditions at $\theta = \theta_1$ and $\theta = \theta_2$ are

$$T_1u = T_2v = T_3w = M_1\psi_x = M_2\psi_\theta = 0. \qquad (17)$$

For boundaries at $x = x_1$ and $x = x_2$ they are

$$T_4 u = T_5 v = T_6 w = M_3 \psi_x = M_4 \psi_\theta = 0. \tag{18}$$

2.2. *Solutions of the equations of motion for freely supported ends*

Consider a noncircular cylindrical shell having its curved ends supported by shear diaphragms (or freely supported). That is, the conditions to be imposed at the ends are

$$T_4 = v = w = M_3 = \psi_\theta = 0 \tag{19}$$

which satisfy eqns (18). Equations (19) are exactly satisfied at $x = 0$ and μ by choosing

$$\begin{aligned} \{u, \psi_x\} &= \sum_{m=1}^{\infty} \{u_m(\theta), \psi_{xm}(\theta)\} \cos \frac{m\pi}{\mu} x \\ \{v, \psi_\theta, w\} &= \sum_{m=1}^{\infty} \{v_m(\theta), \psi_{\theta m}(\theta), w_m(\theta)\} \sin \frac{m\pi}{\mu} x \end{aligned} \tag{20}$$

where m is an integer. Substituting the displacements (20) into the equations of motion (16) yields the following set of ordinary differential equations:

$$\begin{aligned} \alpha_0^4 H[u_m/\Phi - (GH^2/\beta_0)\psi_{xm}] + kT_{4m} + \frac{\mathrm{d}T_{1m}}{\mathrm{d}\theta} &= 0 \\ \alpha_0^4 H[v_m - (GH^2\Phi/\beta_0)\psi_{\theta m}] - \Phi\left(T_{3m} - \frac{\mathrm{d}T_{2m}}{\mathrm{d}\theta} + kT_{5m}\right) &= 0 \\ \alpha_0^4 H w_m + \Phi\left(T_{2m} + \frac{\mathrm{d}T_{3m}}{\mathrm{d}\theta} - kT_{6m}\right) &= 0 \\ \frac{\alpha_0^4}{\beta_0} H^3(\psi_{xm} - G\Phi u_m) + \Phi\left(kM_{3m} - T_{6m} + \frac{\mathrm{d}M_{1m}}{\mathrm{d}\theta}\right) &= 0 \\ \frac{\alpha_0^4}{\beta_0} H^3(\psi_{\theta m}/\Phi - Gv_m) - T_{3m}/G\Phi + \frac{\mathrm{d}M_{2m}}{\mathrm{d}\theta} - kM_{4m} &= 0 \end{aligned} \tag{21}$$

where

$$k = m\pi/\mu \tag{22}$$

and $T_{1m}, T_{2m}, \ldots, M_{4m}$ are expressions obtained by substituting eqns (20) into expressions $T_1, T_2, \ldots, M_4$ in eqns (14) and omitting $\sin kx$ or $\cos kx$. Exact solutions to eqns (21) may be obtained by expressing the ellipticity function (Φ) and the thickness functions ($H, H^2, \ldots$) as infinite power series in θ, and assuming solutions for u_m, v_m, w_m, ψ_{xm} and $\psi_{\theta m}$ which are also infinite power series in θ. The solution procedure will be demonstrated here for shells having cross-sections with one or more axes of symmetry, although it can also be carried out when no symmetry is present.

Let $\theta = 0$ correspond to a symmetry axis. Then the variable coefficients of eqns (21) may be expanded as

$$\begin{aligned} &\{\Phi^2, H, H^2, H^3, H\Phi^2, H^2\Phi^2, H^3\Phi^2\} \\ &\quad = \sum_{n=0}^{\infty} \{\Phi_0\eta_n, A_n^*, B_n^*, C_n^*, \Phi_0 D_n^*, \Phi_0 E_n^*, \Phi_0 F_n^*\}\theta^{2n} \left\{H\Phi\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}, H^3\Phi\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}\right\} \\ &\quad = \sum_{n=0}^{\infty} \Phi_0\{G_n^*, H_n^*\}\theta^{2n-1} \end{aligned} \tag{23}$$

$$\{B_n^*, C_n^*, D_n^*, E_n^*, F_n^*, G_n^*, H_n^*\} = \sum_{p=0}^{n} \{A_{n-p}^* A_p^*, A_{n-p}^* B_p^*,$$

$$A_{n-p}^* \eta_p, B_{n-p}^* \eta_p, C_{n-p}^* \eta_p, (n+1-p)\eta_{n+1-p} A_p^*, (n+1-p)\eta_{n+1-p} C_p^*\}. \quad (24)$$

One can denote Φ^2 as in eqn (23) for the curves for which the curvatures are expressed as even functions of θ. Equations (21) have two solutions: one in which u_m/Φ, w_m, ψ_{xm} are even functions of θ and v_m, $\psi_{\theta m}/\Phi$ are odd functions of θ, and another in which u_m/Φ, w_m, ψ_{xm} are odd functions of θ and v_m, $\psi_{\theta m}/\Phi$ are even functions of θ. They are considered separately.

(i) In the case where u_m/Φ is an even function of θ, one takes:

$$u_m/\Phi = \sum_{n=0}^{\infty} A_n \theta^{2n}, \quad v_m = \sum_{n=0}^{\infty} B_n \theta^{2n+1}, \quad w_m = \sum_{n=0}^{\infty} C_n \theta^{2n}$$

$$\psi_{xm} = \sum_{n=0}^{\infty} D_n \theta^{2n}, \quad \psi_{\theta m}/\Phi = \sum_{n=0}^{\infty} E_n \theta^{2n+1} \quad (25)$$

where A_n, B_n, C_n, D_n and E_n are coefficients which are determined in turn as follows. Substituting eqns (25) into eqns (21) yields

$$\sum_{n=0}^{\infty} (\zeta G^2\Phi_0(2n+1)(2n+2)[\eta_0(G^2\Phi_0 F_0^* + \beta_0 A_0^*)A_{n+1} + GF_0^* D_{n+1}] + f_{na})\theta^{2n} = 0$$

$$\sum_{n=0}^{\infty} (G^2\Phi_0(2n+2)(2n+3)[D_0^*(\beta_0 + G^2\Phi_0 E_0^*)B_{n+1} + G\Phi_0\eta_0 F_0^* E_{n+1}] + f_{nb})\theta^{2n+1} = 0$$

$$\sum_{n=0}^{\infty} (k'\zeta G^2\Phi_0 D_0^*(\beta_0 + G^2\Phi_0 E_0^*)(2n+1)(2n+2)C_{n+1} + f_{nc})\theta^{2n} = 0$$

$$\sum_{n=0}^{\infty} (\zeta G^2\Phi_0 F_0^*(2n+1)(2n+2)(G\Phi_0\eta_0 A_{n+1} + D_{n+1}) + f_{nd})\theta^{2n} = 0$$

$$\sum_{n=0}^{\infty} (G^2\Phi_0\eta_0 C_0^*(2n+2)(2n+3)(GB_{n+1} + E_{n+1}) + f_{ne})\theta^{2n+1} = 0 \quad (26)$$

where f_{na}, f_{nb}, f_{nc}, f_{nd} and f_{ne} are series defined in the Appendix.

(ii) In the case where u_m/Φ is an odd function of θ, one takes:

$$u_m/\Phi = \sum_{n=0}^{\infty} A_n \theta^{2n+1}, \quad v_m = \sum_{n=0}^{\infty} B_n \theta^{2n}, \quad w_m = \sum_{n=0}^{\infty} C_n \theta^{2n+1}$$

$$\psi_{xm} = \sum_{n=0}^{\infty} D_n \theta^{2n+1}, \quad \psi_{\theta m}/\Phi = \sum_{n=0}^{\infty} E_n \theta^{2n} \quad (27)$$

where A_n, B_n, C_n, D_n and E_n are undetermined coefficients. Substituting eqns (27) into eqns (21) yields

$$\sum_{n=0}^{\infty} (\zeta G^2\Phi_0(2n+2)(2n+3)[\eta_0(\beta_0 A_0^* + G^2\Phi_0 F_0^*)A_{n+1} + GF_0^* D_{n+1}] + f_{na})\theta^{2n+1} = 0$$

$$\sum_{n=0}^{\infty} (G^2\Phi_0(2n+1)(2n+2)[D_0^*(\beta_0 + G^2\Phi_0 E_0^*)B_{n+1} + G\Phi_0\eta_0 F_0^* E_{n+1}] + f_{nb})\theta^{2n} = 0$$

$$\sum_{n=0}^{\infty} (k'\zeta G^2\Phi_0 D_0^*(\beta_0 + G^2\Phi_0 E_0^*)(2n+2)(2n+3)C_{n+1} + f_{nc})\theta^{2n+1} = 0$$

$$\sum_{n=0}^{\infty} (\zeta G^2\Phi_0 F_0^*(2n+2)(2n+3)[G\Phi_0\eta_0 A_{n+1} + D_{n+1}] + f_{nd})\theta^{2n+1} = 0$$

$$\sum_{n=0}^{\infty} (G^2\Phi_0\eta_0 C_0^*(2n+1)(2n+2)(GB_{n+1} + E_{n+1}) + F_{ne})\theta^{2n} = 0 \quad (28)$$

where f_{na}, f_{nb}, f_{nc}, f_{nd} and f_{ne} are given in the Appendix. From each set of eqns (26) and (28) used independently, the coefficients A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} and E_{n+1} $(n \geqslant 0)$ are obtained in terms of A_0, B_0, C_0, D_0 and E_0, with the last five left undetermined. Hence, five independent solutions arise from each set. In this way, 10 independent solutions for the complete problem are obtained. The general solutions of eqns (21) are expressed by combining linearly these 10 independent solutions. However, when the cross-section is symmetric about $\theta = 0$, the vibrations are divided into modes which are either symmetric or antisymmetric with respect to the axis of symmetry passing through the axis $\theta = 0$. Then the displacements given by eqns (25) are the solutions for the symmetric modes of vibration, and those from eqns (27) are for the antisymmetric modes.

3. THIN SHELL THEORY (CLASSICAL THEORY)

In thin shell theory, the effects of shear deformation and rotary inertia are neglected. Setting $\gamma_{zx} = \gamma_{\theta z} = 0$ in eqns (6), one finds

$$\psi_x = -\frac{\partial w}{\partial x}, \quad \psi_\theta = -\frac{r_0}{\rho}\left(v + \frac{\partial w}{\partial \theta}\right). \tag{29}$$

Substituting eqn (29) into ε_x, ε_θ and $\gamma_{x\theta}$ in eqns (6) yields

$$\varepsilon_x = \frac{1}{r_0}\left(\frac{\partial u}{\partial x} - \frac{z}{r_0}\frac{\partial^2 w}{\partial x^2}\right)\sin \omega t,$$

$$\varepsilon_\theta = \frac{1}{\rho - z}\left[\frac{\partial v}{\partial \theta} - w - z\frac{\partial}{\partial \theta}\left\{\frac{1}{\rho}\left(\frac{\partial w}{\partial \theta} + v\right)\right\}\right]\sin \omega t,$$

$$\gamma_{x\theta} = \left\{\frac{1}{\rho - z}\frac{\partial u}{\partial \theta} + \frac{1}{r_0}\left(1 - \frac{z}{\rho}\right)\frac{\partial v}{\partial x} - \frac{z}{r_0}\left(\frac{1}{\rho} + \frac{1}{\rho - z}\right)\frac{\partial^2 w}{\partial x\, \partial \theta}\right\}\sin \omega t. \tag{30}$$

These strains are identical to those for the thin shell theory derived by Suzuki *et al.* (1983). Substituting eqns (30) into eqns (10) and neglecting the terms of rotary inertia corresponding to ψ_x and ψ_θ, one obtains the Lagrangian for the thin shell theory that corresponds with that derived previously by the present authors (Suzuki and Leissa, 1985).

4. NUMERICAL CALCULATIONS

Numerical calculations are made for elliptical cylindrical shells of variable thickness. For this type of curvature, Φ_0 and η_n in eqns (23) are

$$\Phi_0 = (1 + \mu_0)^3, \quad \eta_0 = 1,$$

$$\eta_n = \frac{3\mu_0^3(-1)^n 2^{2n}}{(1+\mu_0)^3(2n)!}\left[\frac{1 + 3^{2n-1}}{4} + \frac{2^{2n-1}}{\mu_0} + \frac{1}{\mu_0^2}\right]. \tag{31}$$

To be specific, the following shell parameters are used: $\mu_0 = 0.2$ ($\sim a/b = 1.22$), $\beta_0 = 500$ ($r_0/h_0 = 6.5$) or 1000 ($r_0/h_0 = 9.1$), $v = 0.3$ and $k' = \pi^2/12$, which is the shear coefficient used by Mirsky and Herrmann (1957) and Mirsky (1964) for circular thick cylindrical shells. The cross-section is symmetric with respect to $\theta = 0$, π, and the thickness variation $H(\theta)$ in eqn (23) is taken as

$$H(\theta) = 1 + \varepsilon\theta^2 \tag{32}$$

where ε is an arbitrary constant. In the calculations, three cases where $\varepsilon = 0$, 0.2, 0.4 were considered. These yield ratios of thickness at $\theta = \pi/2$ to that at $\theta = 0$ which are 1, 1.493

and 1.987. The axes of symmetry are the ξ_1-axis passing through the points $\theta = 0, \pi$ and the ξ_2-axis passing through the points $\theta = \pm\pi/2$. Vibrations are divided into four symmetry classes (S–S, S–A, A–S, A–A), depending on whether they are symmetric (S) or antisymmetric (A) with respect to the ξ_1- and ξ_2-axes, respectively. These symmetry classes are obtained by utilizing the symmetric functions (25) or the antisymmetric functions (27), and by enforcing the conditions at $\theta = \pi/2$ that either

$$T_1 = v = T_3 = M_1 = \psi_\theta = 0 \quad \text{(symmetric)} \tag{33}$$

or

$$u = T_2 = w = \psi_x = M_2 = 0 \quad \text{(antisymmetric).} \tag{34}$$

In this work, only the frequency curves for (S–S) and (A–S) are shown because the (A–A) curves are very similar to the (S–S) ones, and the (S–A) curves are also very similar to the (A–S) ones, as found in Suzuki and Leissa (1985).

The displacement functions u_m, v_m, w_m, ψ_{xm} and $\psi_{\theta m}$ were calculated by retaining 100 terms for each of the coefficients A_n, B_n, C_n, D_n and E_n in eqns (25) and (27). Each of the independent solutions corresponding to each symmetry class was obtained by setting one of A_0, B_0, C_0, D_0 and E_0 equal to unity and the others equal to zero. The rate of convergence of the solutions varies with parameters such as β_0, α_0^4/β_0, k, ε and μ_0. In general, the convergence becomes worse as μ_0, β_0, ε, α_0^4/β_0 or k becomes larger. The first three parameters in particular have larger influences on the convergence.

Table 1 shows the convergence of the solutions arising from eqn (25). In the table are shown the significant figures of accuracy of the functions obtained from ($A_0 = 1$, $B_0 = C_0 = D_0 = E_0 = 0$) in the case where $\alpha_0^4/\beta_0 = 0.05$ and $k = 4$. The numbers 75, 66, ... in Table 1 show that 75 or 66 terms are necessary to obtain a function with accuracies of 10 digits, and the numbers (9), (6), ... show that the function converges with accuracies of 9 or 6 digits by 100 terms. For the other four sets of solutions, the same statement can be made. The convergence of the solutions from eqn (27) is similar. The range where the solutions with adequate accuracies can most easily be obtained is $\beta_0 < 1000$, $\mu_0 < 0.4$ and $0 \leqslant \varepsilon < 0.4$.

The general solutions to eqns (21) for symmetric vibration about the ξ_1-axis, for example, are expressed by linearly combining five independent solutions from eqn (25) as follows:

$$\{u_m, v_m, w_m, \psi_{xm}, \psi_{\theta m}\} = \sum_{i=1}^{5} \lambda_i \{u_{mi}, v_{mi}, w_{mi}, \psi_{xmi}, \psi_{\theta mi}\} \tag{35}$$

where $\lambda_1, \ldots, \lambda_5$ are arbitrary constants. Consequently, considering the symmetry conditions of eqns (33) and (34), one may obtain the frequency equations for (S–S) and (S–A) modes in the form of a fifth-order determinant. One can obtain similarly the frequency equations for (A–S) and (A–A) modes using the solutions from eqn (27). The roots of these finite (and relatively small) order frequency determinants are exact values of the nondimensional frequency parameter α_0^4/β_0. The nondimensional frequency parameter α_0^4/β_0 is related to the frequency by

$$\alpha_0^4/\beta_0 = \omega^2 r_0^2 \rho_0 (1-\nu^2)/E. \tag{36}$$

Table 1. Convergence of solutions from eqn (25) ($\alpha_0^4/\beta_0 = 0.05$, $k = 4$)

μ_0	β_0	ε	u_m	v_m	w_m	u_{0m}	v_{0m}	T_{1m}	T_{2m}	T_{3m}	M_{1m}	M_{2m}
		0	75	66	64	69	75	72	78	75	78	76
0.2	500	0.2	86	75	81	82	85	78	89	85	84	74
		0.4	(9)	95	(6)	(8)	(7)	(7)	(8)	85	(8)	(4)
	1000	0.2	98	92	92	94	98	87	90	92	92	91
0.3	500	0	(9)	92	86	95	(9)	95	95	(9)	(9)	(9)

In the calculations for the results shown hereafter, β_0, ε and k were first chosen, and then a search was conducted for the values of α_0^4/β_0 which satisfy the frequency equations.

Figures 2–5 show the comparison between the classical and the improved thick shell theories in nondimensional frequency parameter α_0^4/β_0 versus $k = m\pi r_0/l$ for the (S–S) and (A–S) modes. The curves are for the first, second and third modes of vibration in the order from below, in which the curves are depicted by joining the points for the values of α_0^4/β_0 at $k = 0.5, 1.0, 1.5, \ldots, 4.0$. The numbers $1, \ldots, 12$ on the curves for classical theory in Figs 2 and 3 denote the points at which the mode shapes will be shown later in Figs 8 and 9. The curves for the improved thick shell theory are always below those for the classical theory. The difference between both the theories typically increases as k (and hence r_0/l) becomes large or the vibration mode becomes higher. As seen from Figs 4 and 5, the difference in the case where $\beta_0 = 1000$ becomes smaller than that in the case where $\beta_0 = 500$.

Figures 6 and 7 show the effects of changing ε upon the nondimensional frequency parameter α_0^4/β_0 versus k. The curves become higher with increasing ε.

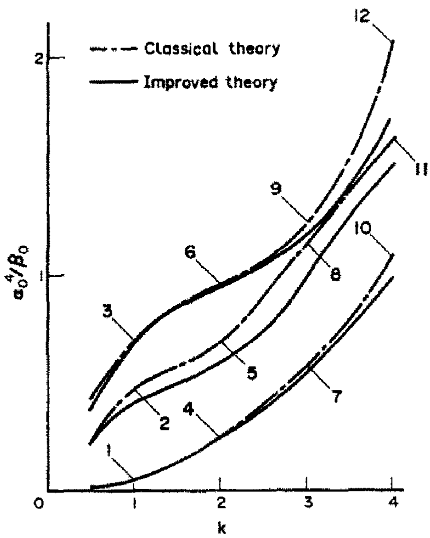

Fig. 2. Frequency curves, (S–S) modes ($\beta_0 = 500$, $\varepsilon = 0$).

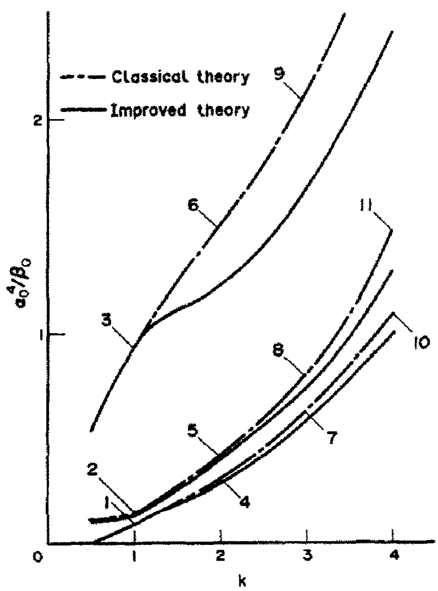

Fig. 3. Frequency curves, (A–S) modes ($\beta_0 = 500$, $\varepsilon = 0$).

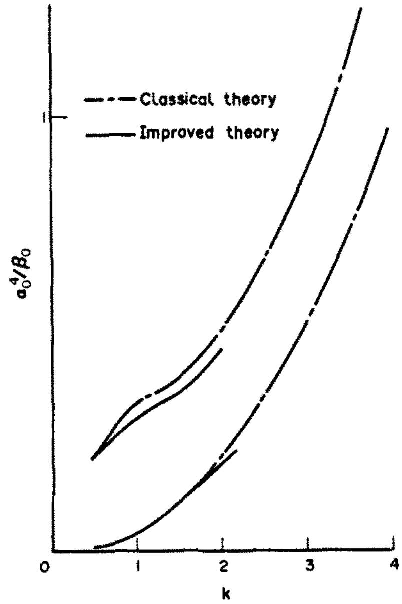

Fig. 4. Frequency curves, (S–S) modes ($\beta_0 = 1000$, $\varepsilon = 0.2$).

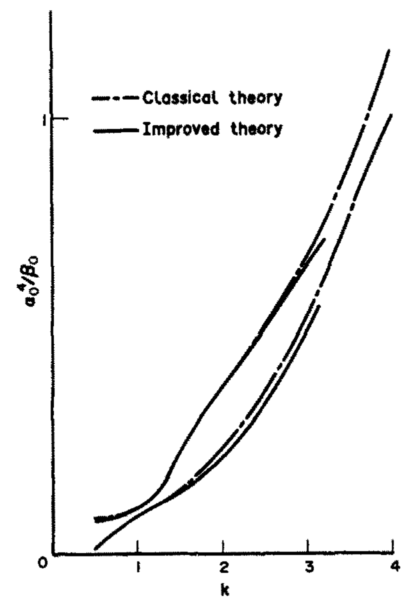

Fig. 5. Frequency curves, (A–S) modes ($\beta_0 = 1000$, $\varepsilon = 0.2$).

Figures 8–11 depict the mode shapes of elliptical cylindrical shells having uniform thickness ($\varepsilon = 0$). They are the displacements w_m for $0 \leqslant \theta \leqslant \pi/2$, in which the maximum amplitude is taken to be unity. Mode numbering corresponds to the numbered points of the frequency curves found in Figs 2, 3, 6 and 7. The mode shapes resulting from both theories are, in most cases, very similar to each other for the lower modes. The mode shapes in the case where $\varepsilon = 0.2, 0.4$, which are not depicted here, are also similar to those of $\varepsilon = 0$.

Table 2 shows the effect of the shear coefficient change upon the relation between α_0^4/β_0 and k. The values of α_0^4/β_0 in the case where $k' = 0.850$ are a little larger in the third decimal place than those where $k' = \pi^2/12$. From this, one finds that the shear coefficient has little effect upon the frequencies.

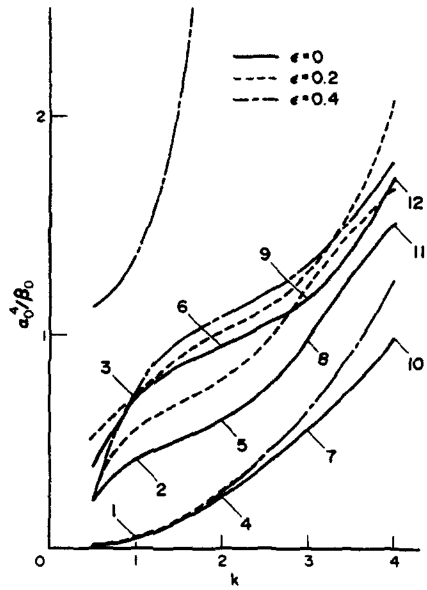

Fig. 6. Frequency curves, (S–S) modes (improved theory, $\beta_0 = 500$).

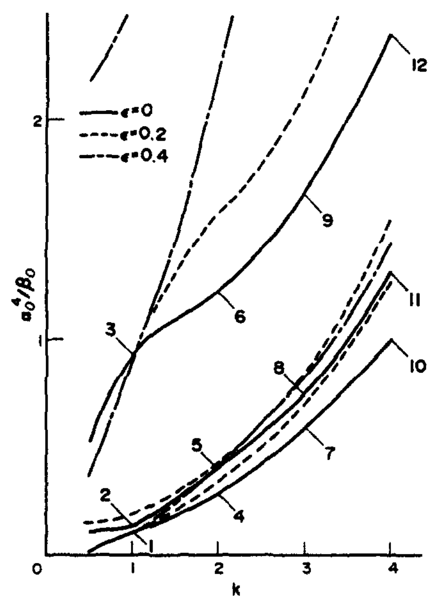

Fig. 7. Frequency curves, (A–S) modes (improved theory, $\beta_0 = 500$).

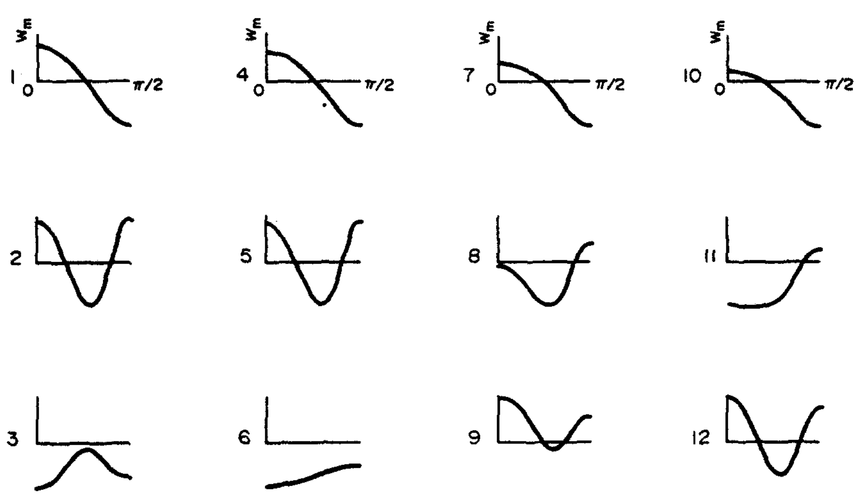

Fig. 8. Mode shapes corresponding to numbered points in Fig. 2 (classical theory).

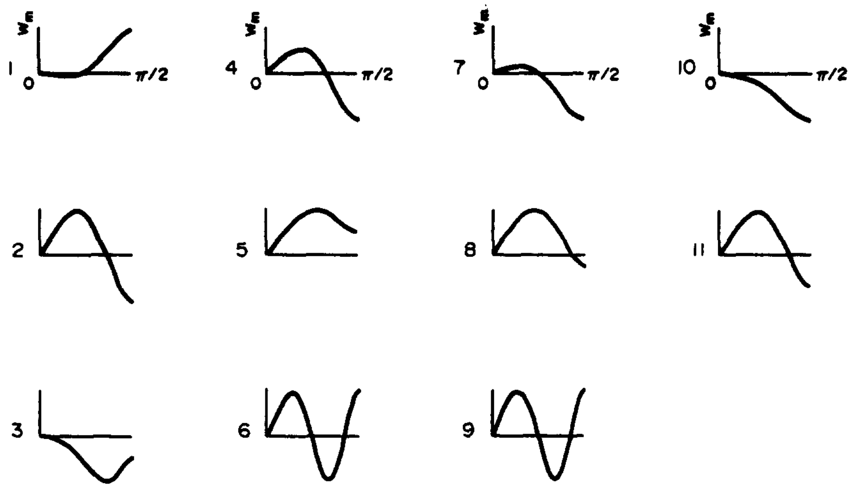

Fig. 9. Mode shapes corresponding to numbered points in Fig. 3 (classical theory).

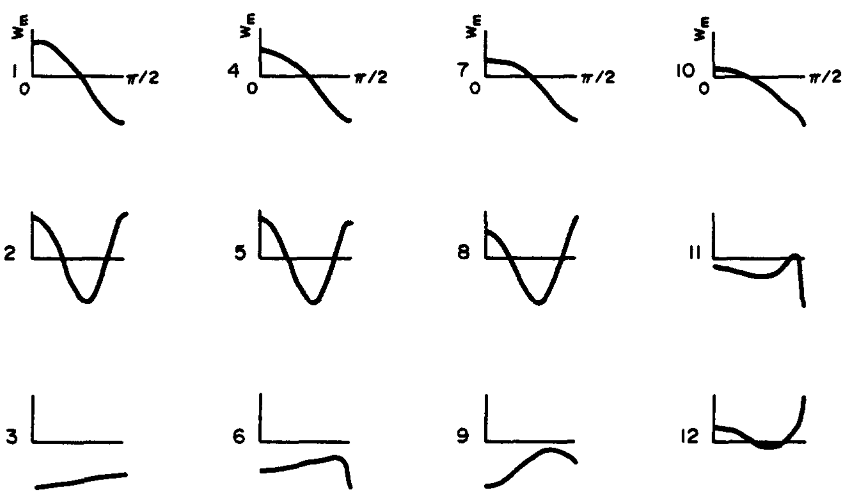

Fig. 10. Mode shapes corresponding to numbered points in Fig. 6 ($\varepsilon = 0$).

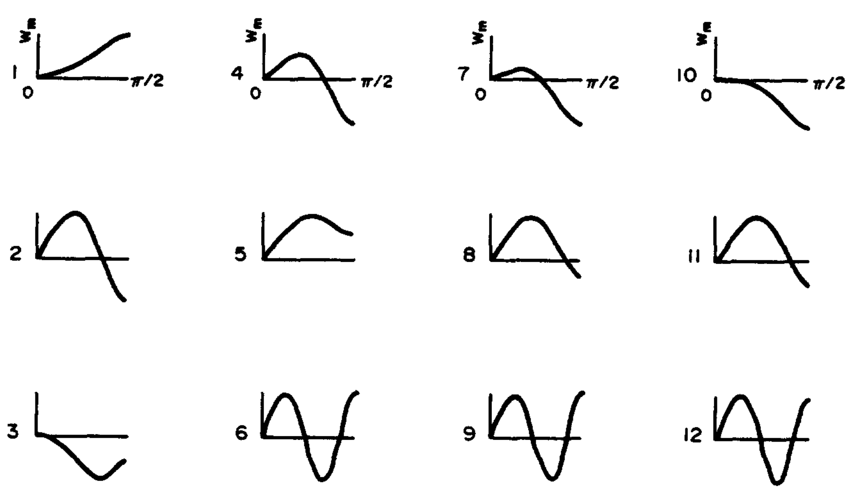

Fig. 11. Mode shapes coprresponding to numbered points in Fig. 7 ($\varepsilon = 0$).

Table 2. Effect of shear coefficient (k') upon α_0^4/β_0 (S–S, $\beta_0 = 500$, $\varepsilon = 0$)

Mode	k'	$k = 1$	$k = 2$	$k = 3$	$k = 4$
1st	0.822	0.049	0.247	0.559	0.984
	0.850	0.049	0.247	0.559	0.986
2nd	0.822	0.420	0.605	0.989	1.512
	0.850	0.421	0.607	0.993	1.515
3rd	0.822	0.702	0.949	1.174	1.713
	0.850	0.702	0.949	1.175	1.719

5. CONCLUSIONS

In this paper, the free vibration of noncircular thick cylindrical shells having circumferential thickness variation has been studied by an improved thick shell theory. The method of solution developed here is a general one applicable to arbitrary noncircular thick cylindrical shells with varying thickness, although limited to shells having freely supported ends. As numerical examples, natural frequencies and mode shapes were found for elliptical cylindrical shells for which the thicknesses vary parabolically in the circumferential direction, and the results were compared with those obtained from thin shell theory. From these results, it is clear that one should use the improved thick shell theory to obtain natural frequencies of noncircular thick cylindrical shells with $\beta_0 < 1000$ because then the shear deformation and rotary inertia have large influences upon the frequencies.

REFERENCES

Leissa, A. W. (1973). *Vibration of Shells.* U.S. Government Printing Office, NASA SP-288.

Love, A. E. H. (1927). *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press.

Mirsky, I. (1964). Vibrations of Orthotropic Thick Cylindrical Shells. *J. Acoust. Soc. Am.* **36**(1), 41–51.

Mirsky, I. and Herrmann, G. (1957). Nonaxially Symmetric Motions of Cylindrical Shells. *J. Acoust. Soc. Am.* **36**(10), 1116–1123.

Suzuki, K. and Leissa, A. W. (1985). Free Vibrations of Noncircular Cylindrical Shells Having Circumferentially Varying Thickness. *J. Appl. Mech.* **52**(1), 149–154.

Suzuki, K., Takahashi, S. and Ishiyama, H. (1978). In-plane Vibrations of Curved Bars. *Bull. JSME* **21**(154), 618–627.

Suzuki, K., Tamura, S., Kosawada, T. and Takahashi, S. (1983). Vibrations of Noncircular Cylindrical Shells. *Bull. JSME* **26**(215), 818–826.

APPENDIX: TERMS CONTAINED IN EQNS (26) AND (28)

For eqn (26)

$$f_{na} = \sum_{p=0}^{n}\left[(\alpha_0^4-k^2\beta_0)A^*_{n-p}A_p+k\beta_0GA^*_{n-p}\{(2n+1)\zeta+(1-2\zeta)(2p+1)\}B_p-k\beta_0G(1-2\zeta)A^*_{n-p}C_p-G\frac{\alpha_0^4}{\beta_0}C^*_{n-p}D_p\right]$$

$$+\zeta G^2\Phi_0(2n+1)\sum_{p=0}^{n-1}(2p+2)\{\eta_0(G^2\Phi_0F^*_{n-p}+\beta_0A^*_{n-p})A_{p+1}+GF^*_{n-p}D_{p+1}\}$$

$$+\sum_{q=0}^{n}\left[k^2GA^*_{n-q}\sum_{p=0}^{q}B^*_{q-p}D_p+\zeta G^2\Phi_0(2n+1)(G^2\Phi_0F^*_{n-q}+\beta_0A^*_{n-q})\sum_{p=0}^{q}(q+p+1)\eta_{q+1-p}A_p\right],$$

$$f_{nb} = -kG\Phi_0\beta_0(1-2\zeta)D^*_0(2n+2)A_{n+1}-G^2\Phi_0D^*_0(2n+2)(\beta_0+G^2\Phi_0E^*_0)C_{n+1}$$

$$+\sum_{p=0}^{n}\Bigg[(\alpha_0^4-\beta_0\zeta k^2)A^*_{n-p}B_p-G\Phi_0\left(\frac{\alpha_0^4}{\beta_0}F^*_{n-p}+k'\beta_0\zeta D^*_{n-p}\right)E_p-k'\beta_0\zeta G^2\Phi_0D^*_{n-p}\{B_p+(2p+2)C_{p+1}\}$$

$$+G^3\Phi_0^2(2n+2)[GD^*_0E^*_{n+1-p}\{(2p+1)B_p-C_p\}+(n+p+2)\eta_{n+1-p}\cdot F^4_0E_p]$$

$$+\beta_0G\Phi_0\{(2n+2)D^*_{n+1-p}-G^*_{n-p}\}[-k(1-2\zeta)A_p+G\{(2p+1)B_p-C_p\}]\Bigg]$$

$$+\sum_{q=0}^{n}\Bigg[-\beta_0k\zeta G\Phi_0A^*_{n-q}\sum_{p=0}^{q+1}(q+p+1)\eta_{q+1-p}\cdot A_p+k^2\zeta G\Phi_0A^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}E_p$$

$$-k'\zeta G^3\Phi_0^2 D^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+2)C_{p+1}\}]$$

$$+G^4\Phi_0^2\{(2n+2)D^*_{n+1-q}-G^*_{n-q}\}\sum_{p=0}^{q}E^*_{q-p}\{(2p+1)B_p-C_p\}$$

$$\left.+G^3\Phi_0^2\{(2n+2)F^*_{n+1-q}-H^*_{n-q}\}\sum_{p=0}^{q}(q+p+1)\eta_{q-p}\cdot E_p\right],$$

$$f_{nc}=k'\zeta G\Phi_0 D_0^*(\beta_0+G^2\Phi_0E_0^*)(2n+1)(GB_n+E_n)$$

$$+\sum_{p=0}^{n}[(\alpha_0^4-k^2k'\beta_0\zeta)A^*_{n-p}C_p-kk'\beta_0\zeta A^*_{n-p}D_p+\beta_0G\Phi_0D^*_{n-p}[-k(1-2\zeta)A_p+G\{(2p+1)B_p-C_p\}]]$$

$$+k'\zeta G\Phi_0\sum_{p=0}^{n-1}[[\beta_0\{(2n+1)D^*_{n-p}-G^*_{n-1-p}\}+G^2\Phi_0D_0^*(2n+1)E^*_{n-p}][G\{B_p+(2p+2)C_{p+1}\}+E_p]]$$

$$+G^3\Phi_0^2\sum_{q=0}^{n}\left[F^*_{n-q}\sum_{p=0}^{q}(q+p+1)\eta_{q-p}E_p+GD^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}\{(2p+1)B_p-C_p\}\right]$$

$$+k'\zeta G^3\Phi_0^2\sum_{q=0}^{n-1}\left[\{(2n+1)D^*_{n-q}-G^*_{n-1-q}\}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+2)C_{p+1}\}]\right],$$

$$f_{nd}=k\zeta G\Phi_0F_0^*(2n+1)E_n+\sum_{p=0}^{n}\left[\left\{C^*_{n-p}\left(\frac{\alpha_0^4}{\beta_0}-k^2\right)-k'\beta_0\zeta A^*_{n-p}\right\}D_p-\frac{\alpha_0^4}{\beta_0}G\Phi_0F^*_{n-p}A_p\right.$$

$$\left.-kk'\beta_0\zeta A^*_{n-p}C_p+\zeta G^3\Phi_0^2F_0^*(2n+1)(n+p+1)\eta_{n+1-p}A_p\right]$$

$$+\zeta G\Phi_0\sum_{p=0}^{n-1}\{(2n+1)F^*_{n-p}-H^*_{n-1-p}\}[G(2p+2)D_{p+1}+kE_p]$$

$$+kG\Phi_0\sum_{q=0}^{n}C^*_{n-q}\sum_{p=0}^{q}\eta_{q-p}\{(1-2\zeta)(q+p+1)E_p+kA_p\}$$

$$+\zeta G^3\Phi_0^2\sum_{q=0}^{n-1}\{(2n+1)F^*_{n-q}-H^*_{n-1-q}\}\sum_{p=0}^{q+1}(q+p+1)\eta_{q+1-p}A_p,$$

$$f_{ne}=-GC_0^*(2n+2)\{G^2\Phi_0\eta_0C_{n+1}+k(1-2\zeta)D_{n+1}\}$$

$$+\sum_{p=0}^{n}\left[GC^*_{n-p}\left(k^2\zeta-\frac{\alpha_0^4}{\beta_0}\right)B_p-C^*_{n-p}\left(k^2\zeta-\frac{\alpha_0^4}{\beta_0}\right)E_p-k\zeta GC^*_{n-p}(2p+2)D_{p+1}\right.$$

$$-k'\beta_0\zeta A^*_{n-p}[E_p+G\{B_p+(2p+2)C_{p+1}\}]-kG(1-2\zeta)(2n+2)C^*_{n+1-p}D_p$$

$$\left.+G^2\Phi_0C_0^*(2n+2)\eta_{n+1-p}[(n+p+2)E_p+G\{(2p+1)B_p-C_p\}]\right]$$

$$+\sum_{q=0}^{n}\left[-k'\zeta G^2\Phi_0A^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+2)C_{p+1}\}]\right.$$

$$\left.+G^2\Phi_0(2n+2)C^*_{n+1-q}\sum_{p=0}^{q}\eta_{q-p}[(q+p+1)E_p+G\{(2p+1)B_p-C_p\}]\right]$$

For eqn (28)

$$f_{na}=k\zeta G\beta_0A_0^*(2n+2)B_{n+1}+\sum_{p=0}^{n}\left[(\alpha_0^4-k^2\beta_0)A^*_{n-p}A_p+kG\beta_0(1-2\zeta)A^*_{n-p}\{(2p+2)B_{p+1}-C_p\}\right.$$

$$-G\frac{\alpha_0^4}{\beta_0}C^*_{n-p}D_p+\zeta G(2n+2)[G\Phi_0(\beta_0A_0^*+G^2\Phi_0F_0^*)(n+p+2)\eta_{n+1-p}A_p$$

$$\left.+k\beta_0A^*_{n+1-p}B_p+G^2\Phi_0(2p+1)F^*_{n+1-p}D_p]\right]$$

$$+\sum_{q=0}^{n}\left[k^2GA^*_{n-q}\sum_{p=0}^{q}B^*_{q-p}D_p+\zeta G^2\Phi_0(2n+2)(\beta_0A^*_{n+1-q}+G^2\Phi_0F^*_{n+1-q})\sum_{p=0}^{q}(q+p+1)\eta_{q-p}A_p\right].$$

$$f_{nb}=-k\beta_0G\Phi_0D_0^*(1-2\zeta)(2n+1)A_n-G^2\Phi_0D_0^*(2n+1)(\beta_0+G^2\Phi_0E_0^*)C_n$$

$$+\sum_{p=0}^{n}\left[(\alpha_0^4-k^2\beta_0\zeta)A^*_{n-p}B_p-G\Phi_0\left(\frac{\alpha_0^4}{\beta_0}F^*_{n-p}+k'\beta_0\zeta D^*_{n-p}\right)E_p-k'\zeta G^2\Phi_0\beta_0D^*_{n-p}\{B_p+(2p+1)C_p\}\right.$$

$$\left.+G^3\Phi_0^2F_0^*(2n+1)(n+p+1)\eta_{n+1-p}E_p\right]$$

$$+G\Phi_0\sum_{p=0}^{n-1}\Bigg[\{(2n+1)D^*_{n-p}-G^*_{n-1-p}\}[-k\beta_0(1-2\zeta)A_p+G(\beta_0+G^2\Phi_0E^*_0)\{(2p+2)B_{p+1}-C_p\}]$$

$$+G^2\Phi_0\eta_0\{(2n+1)F^*_{n-p}-H^*_{n-1-p}\}(2p+2)E_{p+1}+G^3\Phi_0D^*_0(2n+1)E^*_{n-p}\{(2p+2)B_{p+1}-C_p\}\Bigg]$$

$$-\zeta G\Phi_0\sum_{q=0}^{n}\Bigg[k'G^2\Phi_0D^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+1)C_p\}]$$

$$+kA^*_{n-q}\sum_{p=0}^{q}[\beta_0(q+p+1)\eta_{q-p}A_p-kE^*_{q-p}E_p]\Bigg]$$

$$+G^3\Phi_0^2\sum_{q=0}^{n-1}\Bigg[G\{(2n+1)D^*_{n-q}-G^*_{n-1-q}\}\sum_{p=0}^{q-1}E^*_{q-p}\{(2p+2)B_{p+1}-C_p\}+\{(2n+1)F^*_{n-q}-H^*_{n-1-q}\}$$

$$\times\sum_{p=0}^{q}(q+p+1)\eta_{q+1-p}E_p\Bigg],$$

$$f_{nc}=k'\zeta G\Phi_0D^*_0(\beta_0+G^2\Phi_0E^*_0)(2n+2)(GB_{n+1}+E_{n+1})$$

$$+\sum_{p=0}^{n}\Bigg[(\alpha_0^4-k^2k'\beta_0\zeta)A^*_{n-p}C_p-kk'\beta_0\zeta A^*_{n-p}D_p+G\Phi_0\beta_0D^*_{n-p}[-k(1-2\zeta)A_p$$

$$+G\{(2p+2)B_{p+1}-C_p\}]+k'\zeta G\Phi_0[\beta_0\{(2n+2)D^*_{n+1-p}-G^*_{n-p}\}$$

$$+G^2\Phi_0D^*_0(2n+2)E^*_{n+1-p}][E_p+G\{B_p+(2p+1)C_p\}]\Bigg]$$

$$+G^3\Phi_0^2\sum_{q=0}^{n}\Bigg[GD^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}\{(2p+2)B_{p+1}-C_p\}+F^*_{n-q}\sum_{p=0}^{q+1}(q+p+1)\eta_{q+1-p}\cdot E_p$$

$$+k'\zeta\{(2n+2)D^*_{n+1-q}-G^*_{n-q}\}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+1)C_p\}]\Bigg],$$

$$f_{nd}=k\zeta G\Phi_0F^*_0(2n+2)E_{n+1}+\sum_{p=0}^{n}\Bigg[\left(\frac{\alpha_0^4}{\beta_0}C^*_{n-p}-k^2C^*_{n-p}-k'\beta_0\zeta A^*_{n-p}\right)D_p$$

$$-G\Phi_0\frac{\alpha_0^4}{\beta_0}F^*_{n-p}A_p-kk'\beta_0\zeta A^*_{n-p}C_p+\zeta G\Phi_0\{(2n+2)F^*_{n+1-p}-H^*_{n-p}\}\{G(2p+1)D_p+kE_p\}$$

$$+\zeta G^3\Phi_0^2F^*_0(2n+2)(n+p+2)\eta_{n+1-p}A_p\Bigg]$$

$$+\sum_{q=0}^{n}\Bigg[kG\Phi_0C^*_{n-q}\Bigg[(1-2\zeta)\sum_{p=0}^{q+1}(q+p+1)\eta_{q+1-p}E_p+k\sum_{p=0}^{q}\eta_{q-p}A_p\Bigg]$$

$$+\zeta G^3\Phi_0^2\{(2n+2)F^*_{n+1-q}-H^*_{n-q}\}\cdot\sum_{p=0}^{q}(q+p+1)\eta_{q-p}A_p\Bigg],$$

$$f_{ne}=-G^3\Phi_0\eta_0C^*_0(2n+1)C_n+\sum_{p=0}^{n}\Bigg[G\left\{\left(k^2\zeta-\frac{\alpha_0^4}{\beta_0}\right)C^*_{n-p}-k'\beta_0\zeta A^*_{n-p}\right\}B_p$$

$$-k'\beta_0\zeta G(2p+1)A^*_{n-p}C_p-kG\{\zeta(2p+1)+(1-2\zeta)(2n+1)\}C^*_{n-p}D_p$$

$$+\left\{\left(\frac{\alpha_0^4}{\beta_0}-k^2\zeta\right)C^*_{n-p}-k'\beta_0\zeta A^*_{n-p}\right\}E_p\Bigg]+G^2\Phi_0\eta_0(2n+1)\sum_{p=0}^{n-1}C^*_{n-p}\{(2p+2)(GB_{p+1}+E_{p+1})-GC_p\}$$

$$+G^2\Phi_0\sum_{q=0}^{n}\Bigg[(2n+1)C^*_{n-q}\Bigg[\sum_{p=0}^{q}(q+p+1)\eta_{q+1-p}E_p+G\sum_{p=0}^{q-1}\eta_{q-p}\{(2p+2)B_{p+1}-C_p\}\Bigg]$$

$$-k'\zeta A^*_{n-q}\sum_{p=0}^{q}E^*_{q-p}[E_p+G\{B_p+(2p+1)C_p\}]\Bigg].$$